Soliton train dynamics in a weakly nonlocal non-Kerr nonlinear medium

2008 J. Phys. A: Math. Theor. 41 315101

(http://iopscience.iop.org/1751-8121/41/31/315101)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 315101 (13pp) doi:10.1088/1751-8113/41/31/315101

Soliton train dynamics in a weakly nonlocal non-Kerr nonlinear medium

Evgeny V Doktorov and Maxim A Molchan

B I Stepanov Institute of Physics, 220072 Minsk, Belarus

E-mail: doktorov@dragon.bas-net.by and m.moltschan@dragon.bas-net.by

Received 19 April 2008, in final form 12 June 2008
Published 4 July 2008
Online at stacks.iop.org/JPhysA/41/315101

Abstract

We analyze chainlike N-soliton dynamics in a weakly nonlocal, essentially nonintegrable system described by the cubic–quintic nonlinear Schrödinger equation. Quintic nonlinearity is not assumed to be small. This system is reduced to a generalized complex Toda chain model. Numerical simulations demonstrate adverse action of both cubic and quintic nonlocal responses, in their own right, on the quasi-equidistant train propagation, with a development of a chaotic regime. From the Toda chain model, we predict a possibility of mutually compensating both types of nonlocality-induced distortion, restoring thereby a deterministic mode of the train propagation in a weakly nonlocal medium. Analytical predictions corroborate well with numerical results.

PACS numbers: 05.45.Yv, 42.65.Tg, 42.81.Dp

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Interactions between solitons (solitary waves) are the key topic to understand the dynamics of a soliton system. For integrable models, soliton collisions are elastic [1]. A more complicated but at present well-studied case of weak interactions between N solitons with nearly equal amplitudes and velocities in these models [2, 3] is reduced to the integrable complex Toda chain with N nodes [4–11]. Nonintegrable systems which cannot be treated as small perturbations of the integrable ones do not admit such a simple picture. Moreover, even for two tail–tail interacting solitons of the generalized nonlinear Schrödinger (NLS) equation, rich fractal structures in scattering of solitons were revealed [12–16]. It is remarkable that such highly non-regular dynamics can still be successfully described analytically for an arbitrary algebraic (local) nonlinearity in the generalized NLS equation [17, 18].

Our paper is aimed at developing an analytical approach to a description of weakly interacting solitons, arranging into an N-soliton train, in an *essentially nonintegrable* system.

It is important that a generalization from two to N solitons is nontrivial because of lack of the superposition principle for nonlinear dynamical systems. We will not seek for fractals in the soliton interaction. Our goal is rather opposite—to weaken a manifestation of chaotic dynamics of the N-soliton train at the cost of a proper choice of the soliton and medium parameters. The model we shall deal with is the NLS equation with a weakly *nonlocal* cubic–quintic (CQ) nonlinearity. Nonlocality plays a significant part in systems where transport phenomena and finite-range interaction cannot be neglected. A number of new interesting effects attributed to nonlocal nonlinear response have been discovered in nonlinear optics [19, 20], liquid crystals [21, 22] and Bose–Einstein condensates [23–25]. As a particular implementation of our model, pulse propagation in liquid core optical fibers can be mentioned [26, 27]. In the spatial soliton context, our model describes light beam interaction in media of the above class.

Weak nonlocality can be reduced to a perturbation of a basic equation [28, 29] which is *nonintegrable* in our case because we do not assume smallness of the quintic nonlinearity. We prove that the N-soliton train dynamics can be described in terms of a generalized (nonintegrable) complex Toda chain system. We analytically predict a condition to the soliton and medium parameters that can provide a quasi-equidistant regime of the N-soliton train propagation even in a nonlocal medium. Numerical simulations of the CQ NLS equation are found in a very good agreement with predictions made on the basis of the Toda model.

This paper is organized as follows. After formulating a model in section 2, we perform in section 3 a multiple-scale perturbation procedure for deriving evolution equations for parameters of the single CQ NLS soliton. Two-soliton interaction is discussed in section 4. These results are then extended in section 5 to the case of a chainlike N-soliton configuration, and, as a result, we obtain a generalized complex Toda chain model. Section 6 is devoted to numerical simulations. We compare the results which follow from the direct integration of the starting equation, with those found from the Toda model, and prove a good agreement. Section 7 concludes the paper.

2. Model

A model that describes soliton propagation in a nonlocal CQ nonlinear medium is governed by the equation

$$\mathrm{i}u_t + \frac{1}{2}u_{xx} + u\int_{-\infty}^{+\infty} \mathrm{d}x'\, R_3(x-x')|u(x',t)|^2 + \delta u \int_{-\infty}^{+\infty} \mathrm{d}x'\, R_5(x-x')|u(x',t)|^4 = 0. \qquad (1)$$

Here $u(x,t)$ is a complex envelope of the pulse, $R_3(x)$ and $R_5(x)$ are normalized symmetrical response functions of the medium associated with the cubic and quintic nonlinearities, respectively, $\int_{-\infty}^{+\infty} \mathrm{d}x\, R_{3,5}(x) = 1$. For the Gaussian-type nonlocality we have

$$R_{3,5} = \frac{1}{\sqrt{\pi}a_{3,5}} \exp\left(-\frac{x^2}{a_{3,5}^2}\right), \qquad (2)$$

where the positive parameters a_3 and a_5 determine a strength of the nonlocal response of the medium. A similar relation can be written for a more realistic exponential response function [30, 31]. The real parameter δ measures a contribution of the quintic nonlinearity, with $\delta > 0$ standing for the focusing medium, while $\delta < 0$ corresponds to the defocusing medium. It should be stressed once again that we do not assume smallness of δ. Equation (1) is studied numerically in [32], where two types of solitons (fundamental and dipole ones) have been discovered and their stability was analyzed.

In the following, we will be interested in the case of weak nonlocality, when the width of the response function $R(x)$ is finite but small compared to the width of the pulse intensity distribution. This means that intensities in the integrals in (1) can be decomposed as

$$|u(x')|^n = |u(x+(x'-x))|^n = |u(x)|^n + \tfrac{1}{2}(x-x')^2(|u(x)|^n)_{xx} + \mathrm{O}((x-x')^3), \quad n = 2, 4.$$

As a result, in this approximation (1) takes the form of a perturbed CQ NLS equation,

$$\mathrm{i}u_t + \tfrac{1}{2}u_{xx} + |u|^2u + \delta|u|^4u = -\gamma_3 u(|u|^2)_{xx} - \delta\gamma_5 u(|u|^4)_{xx}, \tag{3}$$

where the small positive parameters γ_3 and γ_5 are defined as

$$\gamma_{3,5} = \frac{1}{2}\int \mathrm{d}x\, x^2 R_{3,5}(x).$$

The unperturbed CQ NLS equation, i.e., (3) with $\gamma_3 = \gamma_5 = 0$, admits the exact soliton solution [33–36] for a defocusing quintic nonlinearity ($\delta < 0$):

$$u(x,t) = \Phi(y,\eta)\exp\left(2\mathrm{i}Vy + \mathrm{i}\sigma\right), \tag{4}$$

where a positive parameter η determines the soliton amplitude, V stands for the soliton velocity and σ is a phase. More precisely,

$$\Phi(y,\eta) = 2\eta\left(\frac{2B}{B+\cosh(4\eta y)}\right)^{\frac{1}{2}}, \qquad B = \left(1+\frac{32}{3}\eta^2\delta\right)^{-\frac{1}{2}},$$
$$y = x - 2Vt - x_0, \qquad \sigma = 2(\eta^2+V^2)t - \sigma_0, \tag{5}$$

and the real function Φ satisfies the equation

$$\tfrac{1}{2}\Phi_{yy} - 2\eta^2\Phi + \Phi^3 + \delta\Phi^5 = 0. \tag{6}$$

Hence, for $\delta < 0$ we have $B > 1$ with the natural restriction

$$1 - \tfrac{32}{3}\eta^2|\delta| > 0. \tag{7}$$

Recently [37] the solution (5) was extended to the focusing quintic nonlinearity, and its stability was proved.

As we see, the CQ soliton depends on four real parameters: η, V, soliton maximum position x_0 and phase σ_0. In what follows, we will also need the expressions for the soliton power $P(\eta)$ and the derivative P_η:

$$P(\eta) = \int_{-\infty}^{+\infty}\mathrm{d}y\,\Phi^2(y,\eta) = \frac{4B\eta}{\sqrt{B^2-1}}\ln\frac{\sqrt{B+1}+\sqrt{B-1}}{\sqrt{B+1}-\sqrt{B-1}}, \qquad \delta < 0,$$
$$P(\eta) = \frac{8B\eta}{\sqrt{1-B^2}}\arctan\sqrt{\frac{1-B}{1+B}}, \qquad \delta > 0, \tag{8}$$

and in both cases $P_\eta = 4B^2$.

3. The perturbed CQ NLS equation

We will solve (3) by means of the multiple-scale perturbation theory starting from the unperturbed CQ NLS equation. Let us write (3) as

$$\mathrm{i}u_t + \tfrac{1}{2}u_{xx} + |u|^2u + \delta|u|^4u = \epsilon G(u) \tag{9}$$

with a small perturbation parameter ϵ. The function $G(u)$ gives a functional form of the perturbation. The perturbation induces an evolution of the soliton parameters on the slow time scale $T = \epsilon t$. We seek for the perturbed soliton solution in the form

$$u = \tilde{\Phi}(y,\eta,t,T)\,\mathrm{e}^{\mathrm{i}\varphi}, \qquad \varphi = 2Vy + \sigma, \tag{10}$$

where the coordinate y and phase σ are written as

$$y = x - 2\int_0^t \mathrm{d}t'\, V - x_0, \qquad \sigma = 2\int_0^t \mathrm{d}t'(\eta^2 + V^2) - \sigma_0, \tag{11}$$

and the soliton parameters $\eta(T)$, $V(T)$, $x_0(T)$ and $\sigma_0(T)$ depend on the slow time T. Inserting the ansatz (10) into (9) gives

$$\tilde{\Phi}_t + \tfrac{1}{2}\tilde{\Phi}_{yy} - 2\eta^2\tilde{\Phi} + |\tilde{\Phi}|^2\tilde{\Phi} + \delta|\tilde{\Phi}|^4\tilde{\Phi} = \epsilon G\mathrm{e}^{-\mathrm{i}\varphi} - \mathrm{i}\epsilon\left(\tilde{\Phi}_\eta \eta_T - \tilde{\Phi}_y x_{0_T}\right) - \epsilon\left(2V x_{0_T} - 2V_T y + \sigma_{0_T}\right)\tilde{\Phi}.$$

Following the well-known formalism [38–41], we expand $\tilde{\Phi}$ in a series with respect to ϵ:

$$\tilde{\Phi} = \Phi(y, \eta) + \epsilon\phi + O(\epsilon^2).$$

Evidently, the zero-order term Φ obeys (6). At order ϵ we obtain the equation for ϕ:

$$\phi_t + \tfrac{1}{2}\phi_{yy} - 2\eta^2\phi + \Phi^2(1 + \delta\Phi^2)\phi + \Phi^2(1 - 2\delta\Phi^2)(\phi + \phi^*) = \epsilon G\mathrm{e}^{-\mathrm{i}\varphi} - \mathrm{i}\epsilon\left(\Phi_\eta \eta_T - \Phi_y x_{0_T}\right) - \epsilon\left(2V x_{0_T} - 2V_T y + \sigma_{0_T}\right)\Phi.$$

Combining this equation with the complex conjugate one, we arrive at the linear inhomogeneous equation

$$\mathrm{i}\Psi_t + L\Psi = H, \tag{12}$$

where

$$\Psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}\phi + \phi^*\\ \phi^* - \phi\end{pmatrix}, \qquad L = \begin{pmatrix}0 & L_0\\ L_1 & 0\end{pmatrix}, \qquad H = \begin{pmatrix}H_1\\ H_2\end{pmatrix}$$

and

$$\begin{aligned} L_0 &= -\tfrac{1}{2}\partial_y^2 + 2\eta^2 - \Phi^2 - \delta\Phi^4, \qquad L_1 = L_0 - 2\Phi^2(1 - \delta\Phi^2),\\ H_1 &= 2\mathrm{i}\left[\mathrm{Im}(G\mathrm{e}^{-\mathrm{i}\varphi}) - \Phi_\eta\eta_T + \Phi_y x_{0_T}\right],\\ H_2 &= 2\left[-\mathrm{Re}(G\mathrm{e}^{-\mathrm{i}\varphi}) + \left(2V x_{0_T} - 2V_T y + \sigma_{0_T}\right)\Phi\right]. \end{aligned} \tag{13}$$

The operator L is self-adjoint and has two eigenfunctions with zero eigenvalues, as well as two generalized eigenfunctions [38]:

$$L\Psi_j = 0, \qquad L\widetilde{\Psi}_j = \Psi_j, \qquad j = 1, 2,$$

where

$$\Psi_1 = \begin{pmatrix}\Phi_y\\ 0\end{pmatrix}, \qquad \Psi_2 = \begin{pmatrix}0\\ \Phi\end{pmatrix}, \qquad \widetilde{\Psi}_1 = \begin{pmatrix}0\\ -y\Phi\end{pmatrix}, \qquad \widetilde{\Psi}_2 = \begin{pmatrix}-(1/4\eta)\,\Phi_\eta\\ 0\end{pmatrix}. \tag{14}$$

To avoid secularity development in solutions to (12) for large time, we should impose the orthogonality conditions:

$$\langle H, \Psi_j\rangle = \langle H, \widetilde{\Psi}_j\rangle = 0, \qquad j = 1, 2. \tag{15}$$

Here the inner product is defined as

$$\langle F_1, F_2\rangle = \int_{-\infty}^{+\infty} \mathrm{d}y\, F_1^\dagger(y)\sigma_1 F_2(y),$$

and σ_1 is the Pauli matrix. Accounting for (13) and (14), we derive from (15) the slow evolution equations for the soliton parameters:

$$\begin{aligned} PV_T &= \int \mathrm{d}y\, \Phi_y\, \mathrm{Re}(G\mathrm{e}^{-\mathrm{i}\varphi}), \qquad P_\eta\eta_T = 2\int \mathrm{d}y\, \Phi_\eta\, \mathrm{Im}(G\mathrm{e}^{-\mathrm{i}\varphi}),\\ Px_{0_T} &= 2\int \mathrm{d}y\, y\Phi\, \mathrm{Im}(G\mathrm{e}^{-\mathrm{i}\varphi}), \qquad P_\eta\left(V x_{0_T} + \frac{1}{2}\sigma_{0_T}\right) = \int \mathrm{d}y\, \Phi_\eta\, \mathrm{Re}(G\mathrm{e}^{-\mathrm{i}\varphi}). \end{aligned} \tag{16}$$

These equations will be used below to describe dynamics of weakly interacting CQ solitons. First, we will consider the two-soliton interaction. Then these results will be generalized to N solitons.

4. Two-soliton interaction

We consider two well-separated tail–tail interacting solitons with the (almost) equal velocities and amplitudes. The degree of the tail–tail overlapping is measured by a small parameter $\epsilon = \mathrm{e}^{-2\eta(\xi_2-\xi_1)}$, where $\xi_2 - \xi_1 \gg 1$ is the intersoliton spacing and η is the mean amplitude of solitons. As regards the parameters γ_3 and γ_5, they are supposed to be of the order of $\sqrt{\epsilon}$. Deviations of the individual soliton amplitudes and velocities from their mean values are also of the order of $\sqrt{\epsilon}$. Then, neglecting the effects of the order of $\epsilon^{3/2}$ and smaller, we have the equation for the jth soliton ($j = 1, 2$):

$$\begin{aligned} \mathrm{i}u_{jt} + \tfrac{1}{2}u_{jxx} + |u_j|^2u_j + \delta|u_j|^4u_j &= -2|u_j|^2u_{3-j} - u_j^2u_{3-j}^* - u_j^2u_{3-j}^* \\ &\quad - \delta\left(3|u_j|^4u_{3-j} + 2|u_j|^2u_j^2u_{3-j}^*\right) - \gamma_3u_j(|u_j|^2)_{xx} - \gamma_5\delta u_j(|u_j|^4)_{xx}. \end{aligned} \tag{17}$$

The right-hand side of (17) represents a perturbation experienced by the jth CQ soliton. Here

$$\begin{aligned} u_j &= \Phi_j(y_j,\eta_j)\,\mathrm{e}^{\mathrm{i}\varphi_j}, \qquad y_j = x - \xi_j, \qquad \varphi_j = 2V_jy_j + \sigma_j, \\ \xi_j &= 2\int_0^t \mathrm{d}t'\,V_j + x_{0j}, \qquad \sigma_j = 2\int_0^t \mathrm{d}t'\left(\eta_j^2 + V_j^2\right) - \sigma_{j0}. \end{aligned} \tag{18}$$

Let us first consider an influence of the second (right) soliton on the first (left) one. It follows from (17) and (18) that

$$\mathrm{Re}(\epsilon G_1\,\mathrm{e}^{-\mathrm{i}\varphi_1}) = -\Phi_1^2\Phi_2\left(3 + 5\Phi_1^2\right)\cos\varphi_{21} - \frac{\gamma_3}{8\eta^2B}\Phi_1^7I_3(y_1) - \frac{\gamma_5\delta}{2\eta^2B}\Phi_1^9I_5(y_1),$$

$$\mathrm{Im}(\epsilon G_1\,\mathrm{e}^{-\mathrm{i}\varphi_1}) = -\Phi_1^2\Phi_2\left(1 + \delta\Phi_1^2\right)\sin\varphi_{21}.$$

Here

$$\varphi_{21} = \varphi_2 - \varphi_1 \approx -2V\xi_{21} + \sigma_{21}, \qquad \xi_{21} = \xi_2 - \xi_1, \qquad \sigma_{21} = \sigma_2 - \sigma_1, \tag{19}$$

and we use in (19) the mean velocity $V = \frac{1}{2}(V_1 + V_2)$ because V_1 and V_2 are (almost) the same. Similarly,

$$y_2 = y_1 - (\xi_2 - \xi_1), \qquad \xi_2 - \xi_1 \gg 1,$$

and

$$\Phi_2(y_2,\eta_2) \approx 4\eta\sqrt{B}\exp\left[2\eta(y_1 - \xi_2 + \xi_1)\right], \qquad B = \left(1 + \frac{32}{3}\eta^2\delta\right)^{-\frac{1}{2}}. \tag{20}$$

The presence of a small factor $\exp[-2\eta(\xi_2 - \xi_1)]$ in Φ_2 (20) allows us to use once again the mean amplitude η in B. Finally, the functions $I_3(y_1)$ and $I_5(y_1)$ are found from the explicit form of $\Phi_1(y_1)$:

$$\begin{aligned} I_3(y_1) &= \cosh(8\eta y_1) - 2B\cosh(4\eta y_1) - 3, \\ I_5(y_1) &= \cosh(8\eta y_1) - B\cosh(4\eta y_1) - 2. \end{aligned} \tag{21}$$

Substituting the above relations into (16) and taking into account that $\Phi(y_1)$ and $\Phi_\eta(y_1)$ are even functions of y_1, we obtain (see also [17])

$$P_\eta\eta_{1t} = -64\eta^3B\mathrm{e}^{-2\eta\xi_{21}}\sin\varphi_{21}, \qquad PV_{1t} = 64\eta^4B\mathrm{e}^{-2\eta\xi_{21}}\cos\varphi_{21}. \tag{22}$$

Note the appearance of derivatives in t in (22) due to ϵ in $\mathrm{Re}(\epsilon G_1\,\mathrm{e}^{-\mathrm{i}\varphi_1})$ and $\mathrm{Im}(\epsilon G_1\,\mathrm{e}^{-\mathrm{i}\varphi_1})$. Besides, we use here the mean power $P(\eta)$.

As regards the evolution equation for x_{0t}, it is sufficient to know that

$$x_{0t} = \mathrm{O}(\epsilon). \tag{23}$$

This statement will be elucidated below. Lastly, we are left with the equation

$$P_\eta(\sigma_{01})_t = 2\int \mathrm{d}y_1\, \Phi_\eta(y_1)\mathrm{Re}(G_1\,\mathrm{e}^{-\mathrm{i}\varphi_1}),$$

where we should explicitly calculate the contribution of the terms with γ_3 and γ_5. Hence,

$$P_\eta(\sigma_{01})_t = \frac{1}{4\eta_1^2 B}\int_{-\infty}^{+\infty} \mathrm{d}y_1\, \Phi_\eta(y_1)\Phi^7(y_1)[\gamma_3 I_3(y_1) + 4\gamma_5\delta\Phi^2(y_1)I_5(y_1)] + \mathrm{O}(\epsilon). \tag{24}$$

Calculating the integrals yields

$$(\sigma_{01})_t = \frac{4}{3}\eta_1^4\frac{B}{1-B^2}F(\gamma_3,\gamma_5) + \mathrm{O}(\epsilon), \tag{25}$$

where

$$F(\gamma_3,\gamma_5) = 48\left\{\gamma_3\left(1-\frac{P(\eta)}{4\eta}\right) - \frac{8}{15}\gamma_5\delta\frac{\eta^2}{1-B^2}\left[B^2(44+B^2) - (14+31B^2)\frac{P(\eta)}{4\eta}\right]\right\}. \tag{26}$$

Repeating the above steps for the second soliton, we find the corresponding evolution equations:

$$\begin{aligned} &P_\eta\eta_{2t} = 64\eta^3 B\mathrm{e}^{-2\eta\xi_{21}}\sin\varphi_{21}, \qquad PV_{2t} = -64\eta^4 B\mathrm{e}^{-2\eta\xi_{21}}\cos\varphi_{21},\\ &(x_{20})_t = \mathrm{O}(\epsilon), \qquad (\sigma_{20})_t = \frac{4}{3}\eta_2^4\frac{B}{1-B^2}F(\gamma_3,\gamma_5) + \mathrm{O}(\epsilon). \end{aligned} \tag{27}$$

In the following section we will generalize (22)–(27) to the case of N solitons.

5. *N*-soliton train interaction and generalized Toda chain

To derive evolution equations for the parameters of the jth soliton incorporated into a chainlike configuration of N solitons, we should keep in mind that the interaction force between the solitons is of the order of their overlap. Hence, we take into account only the nearest-neighbor interaction. As in the case of two solitons, we assume that solitons within the train have initially equal or nearly equal velocities and amplitudes. Lastly, the overlap between neighboring solitons is taken to be small. Then, a generalization of the evolution equations (22)–(27) to the case of N solitons is straightforward. For further convenience we will use in calculations ξ_j and σ_j (18) instead of x_{0j} and σ_{0j}, respectively. As a result, the evolution equations we are seeking for are written as follows:

$$\begin{aligned} &P_\eta\eta_{jt} = -64\eta^3 B\big(\mathrm{e}^{-2\eta\xi_{j+1,j}}\sin\varphi_{j+1,j} - \mathrm{e}^{-2\eta\xi_{j,j-1}}\sin\varphi_{j,j-1}\big),\\ &PV_{jt} = 64\eta^4 B\big(\mathrm{e}^{-2\eta\xi_{j+1,j}}\cos\varphi_{j+1,j} - \mathrm{e}^{-2\eta\xi_{j,j-1}}\cos\varphi_{j,j-1}\big),\\ &\xi_{jt} = 2V_j + \mathrm{O}(\epsilon), \qquad \sigma_{jt} = 2\left(\eta_j^2 + V_j^2\right) - \frac{4}{3}\eta_j^4\frac{B}{1-B^2}F(\gamma_3,\gamma_5) + \mathrm{O}(\epsilon). \end{aligned} \tag{28}$$

Here $j = 1,\ldots,N, \xi_{j,j-1} = \xi_j - \xi_{j-1}, \varphi_{j,j-1} = \varphi_j - \varphi_{j-1}$, and we formally put $\exp(-2\eta\xi_{1,0}) = \exp(-2\eta\xi_{N+1,N}) = 0$.

Note that it is a change-over from x_{0j} to ξ_j that justifies the adequacy of the estimation in (23). Besides, the mean values η and V are constants of motion, as should be. Moreover, it is seen from the equation for σ_j that there exists a possibility of mutually compensating contributions of cubic and quintic nonlocalities at the cost of a proper choice of the parameters of solitons and medium.

Let us introduce a complex quantity

$$\lambda_j = \frac{P}{\eta P_\eta} V_j - \mathrm{i}\eta_j$$

and denote $(P/\eta P_\eta) = 1+\alpha$ (see also [17]). We will see below that the parameter α measures a 'departure from integrability' of the local CQ NLS equation. Note that we do not consider α small. Differentiation of λ_j in t with account for (28) yields

$$\frac{\mathrm{d}\lambda_j}{\mathrm{d}t} = \frac{16\eta^3}{B}[\exp(-2\eta\xi_{j+1,j} + \mathrm{i}\varphi_{j+1,j}) - \exp(-2\eta\xi_{j,j-1} + \mathrm{i}\varphi_{j,j-1})]. \tag{29}$$

In terms of $E_{j+1,j}$ defined as

$$E_{j+1,j} = \frac{2\eta^2}{B}\exp(-2\eta\xi_{j+1,j} + \mathrm{i}\varphi_{j+1,j}) = \exp\left[-2\eta\xi_{j+1,j} - 2\mathrm{i}V\xi_{j+1,j} + \mathrm{i}\sigma_{j+1,j} + \ln\frac{2\eta^2}{B}\right] \tag{30}$$

(we use here (19)), we obtain from (29)

$$\frac{\mathrm{d}\lambda_j}{\mathrm{d}t} = 8\eta(E_{j+1,j} - E_{j,j-1}). \tag{31}$$

On the other hand, equations (30) and (28) permit us to write

$$\frac{\mathrm{d}E_{j+1,j}}{\mathrm{d}t} = -4\eta\left[\lambda_{j+1} - \lambda_j - \alpha(V_{j+1} - V_j) + \mathrm{i}\frac{\eta B}{1-B^2}F(\gamma_3,\gamma_5)(\eta_{j+1} - \eta_j)\right]E_{j+1,j}. \tag{32}$$

In the process of derivation of (32) we used

$$V_{j+1}^2 - V_j^2 \approx 2V(V_{j+1} - V_j), \qquad \eta_{j+1}^3 - \eta_j^3 \approx 3\eta^2(\eta_{j+1} - \eta_j).$$

Now we represent $E_{j+1,j}$ as

$$E_{j+1,j} = -\exp(q_{j+1} - q_j). \tag{33}$$

Then it follows from (32) and (33) that

$$\frac{\mathrm{d}q_j}{\mathrm{d}t} = -4\eta\left[\lambda_j - \alpha V_j + \mathrm{i}\frac{\eta B}{1-B^2}F(\gamma_3,\gamma_5)\eta_j\right]. \tag{34}$$

Finally, combining (31)–(34) and using the normalized time $\tau = 4\sqrt{2}\eta t$, we obtain a generalized complex Toda chain model:

$$\frac{\mathrm{d}^2 q_j}{\mathrm{d}\tau^2} = \mathrm{e}^{q_{j+1}-q_j} - \mathrm{e}^{q_j - q_{j-1}} - \frac{\alpha}{1+\alpha}\mathrm{Re}(\mathrm{e}^{q_{j+1}-q_j} - \mathrm{e}^{q_j-q_{j-1}}) - \mathrm{i}\frac{\eta B}{1-B^2}F(\gamma_3,\gamma_5)\mathrm{Im}(\mathrm{e}^{q_{j+1}-q_j} - \mathrm{e}^{q_j-q_{j-1}}). \tag{35}$$

As regards the explicit expression for q_j, it follows from (30) and (33) that q_j has the form

$$q_j = -2\mathrm{i}(V - \mathrm{i}\eta)\xi_j + \mathrm{i}\sigma_j + j\ln(2\eta^2/B) + \mathrm{i}j\pi + \mathrm{i}C(t), \tag{36}$$

where $C(t)$ is a function to be determined. Differentiating q_j (36) in t in virtue of (28) and comparing the result with (34) yields $C(t) = \sigma(t)$, where $\sigma(t)$ is the mean value of phase. Hence,

$$q_j = -2\mathrm{i}(\lambda - \alpha V)\xi_j + \mathrm{i}(\sigma_j + \sigma) + j\ln(2\eta^2/B) + \mathrm{i}j\pi.$$

Here λ is the mean value

$$\lambda = \frac{1}{N}\sum_{j=1}^{N}\lambda_j.$$

Note that the phase variables σ_j and σ depend on the nonlocality parameters γ_3 and γ_5.

Some comments concerning the generalized Toda chain model (35) are quite suitable here. First, for a local CQ medium ($\gamma_3 = \gamma_5 = 0$) we are still left with the nonintegrable Toda model, due to the presence of the quintic-induced term with α. Second, the function q_j (36) is δ-dependent and hence is more general than if we would start with the integrable cubic NLS equation. Third, we can use predictions made on the basis of the integrable complex Toda chain model [6, 7] concerning various dynamical regimes of the cubic NLS soliton train evolution, to estimate a possibility of realizing similar regimes for the CQ NLS solitons in nonlocal media. This subject is discussed in the following section.

6. Comparison with numerical simulations

Let us recall that we started from the perturbed CQ NLS equation (3) and reduced it to the generalized Toda chain model (35) (or, equivalently, to the dynamical system (28)). Such a reduction allows us to predict various dynamical regimes of the N-soliton train evolution, relying on similar results for the integrable Toda chain [7, 8]. Below we perform a comparison of our predictions with direct numerical integration of the starting equation (3).

We study evolution of two sorts of the soliton train composed from three and five CQ NLS solitons, respectively, which propagate in a weakly nonlocal medium. For further references and to demonstrate our conclusions more informatively, we first consider the soliton train propagation in a local CQ NLS model ($\gamma_3 = \gamma_5 = 0$). Motivated by the results on the train dynamics in the integrable Toda model, we take initial soliton amplitudes in the form

$$\eta_1 = \eta + \beta, \qquad \eta_2 = \eta, \qquad \eta_3 = \eta - \beta$$

for three solitons, and

$$\eta_1 = \eta, \qquad \eta_2 = \eta + \beta, \qquad \eta_3 = \eta, \qquad \eta_4 = \eta - \beta, \qquad \eta_5 = \eta$$

for five solitons. Here η is the mean amplitude and β is a small deviation from the mean value. In our simulations we put $\beta = 0.05$, $\delta = \pm 0.1$; the initial separation between adjacent solitons is $\xi_j - \xi_{j-1} = 8$ and initial velocities are taken to be zero. Initial phases of solitons were chosen to be π-alternating, according to the conditions of the quasi-equidistant soliton propagation derived in frames of the integrable complex Toda chain model [7, 10]. Figures 1(*a*) and 2(*a*) illustrate comparison of soliton trajectories ξ_j obtained by solving the dynamical system (28) (dashed lines) with solitons' tracks obtained numerically from the CQ NLS equation (3). The agreement is impressive.

Note a crucial role of the initial amplitude mismatch for the stabilization of the soliton train (a fact well known for the NLS solitons). Almost equidistant propagation of solitons was observed only in the case of the nonzero mismatch whose value was more than eight percent of the mean amplitude. The results turned out to be valid for both signs of δ due to the chosen value of the amplitude mismatch.

In order to elucidate the influence of cubic and quintic nonlocalities on the train dynamics, we integrated (3) separately for $\gamma_3 \neq 0$ and $\gamma_5 = 0$ (cubic nonlocality contribution only) and $\gamma_3 = 0$ and $\gamma_5 \neq 0$ (quintic nonlocality contribution only). As it follows from figures 1(*b*) and 2(*b*), cubic nonlocality tends to destabilize the train, though for five solitons this effect is less pronounced. In the case of quintic nonlocality we observe a much more dramatic

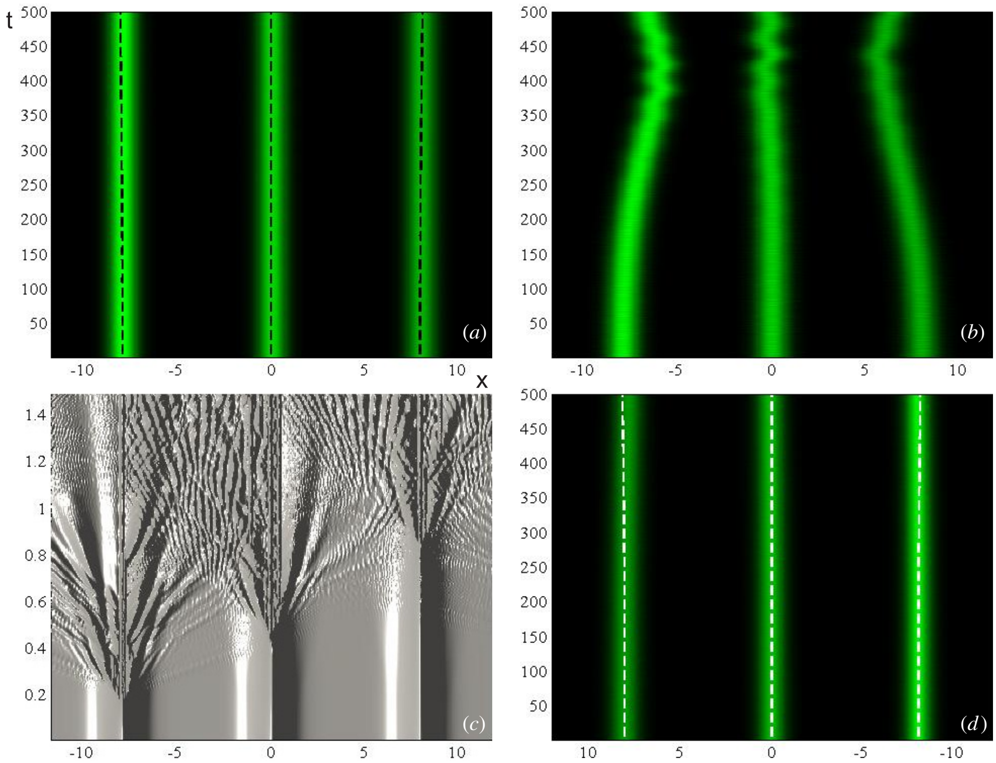

Figure 1. Dynamics of the three-soliton train: (*a*) quasi-equidistant propagation in a local CQ medium ($\gamma_3 = \gamma_5 = 0$), dashed lines correspond to the trajectories ξ_1, ξ_2 and ξ_3 obtained from equations (28), thick lines are solitons' tracks calculated from (3); (*b*) disturbed quasi-equidistant propagation in a medium with only cubic nonlocality ($\gamma_3 = 0.015, \gamma_5 = 0$); (*c*) chaotic disintegration of solitons in a medium with only quintic nonlocality ($\gamma_3 = 0, \gamma_5 = 0.05$); (*d*) restoration of the quasi-equidistant regime for nonlocality parameters obeying the condition (37) ($\gamma_3 = 0.015, \gamma_5 = 0.05$). Initial configuration on all plots: $\beta = 0.05, \eta = \sqrt{2/3}, V_1 = V_2 = V_3 = 0, \varphi_{2,1} = \varphi_{3,2} = \pi, \xi_j - \xi_{j-1} = 8, \delta = -0.1$.

soliton behavior (figures 1(*c*) and 2(*c*)). Defocusing quintic nonlocality leads very quickly to chaotic disintegration of solitons and their eventual decay. Such a situation manifests the nonintegrable origin of the model under consideration and seems quite natural after discovering fractal structures in two weakly interacting soliton systems [17].

Despite the fact that both cubic and quintic nonlocalities, treated separately, disturb the quasi-equidistant soliton train propagation, there exists a unique possibility of mutually compensating adverse effects of nonlocalities. Indeed, we can restore a practically deterministic regime displayed in figures 1(*a*) and 2(*a*) under the definite relation between the soliton and medium parameters. It follows from (28) and (35) that such a compensation is accomplished if the function $F(\gamma_3, \gamma_5)$ (26) vanishes, or, equivalently, if the condition

$$\gamma_3 = \Gamma(\eta, \delta)\gamma_5 \tag{37}$$

takes place, where

$$\Gamma(\eta, \delta) = \frac{8}{15}\frac{\delta\eta^2}{1 - B^2}\frac{4\eta B^2(44 + B^2) - (14 + 31B^2)P(\eta)}{4\eta - P(\eta)}. \tag{38}$$

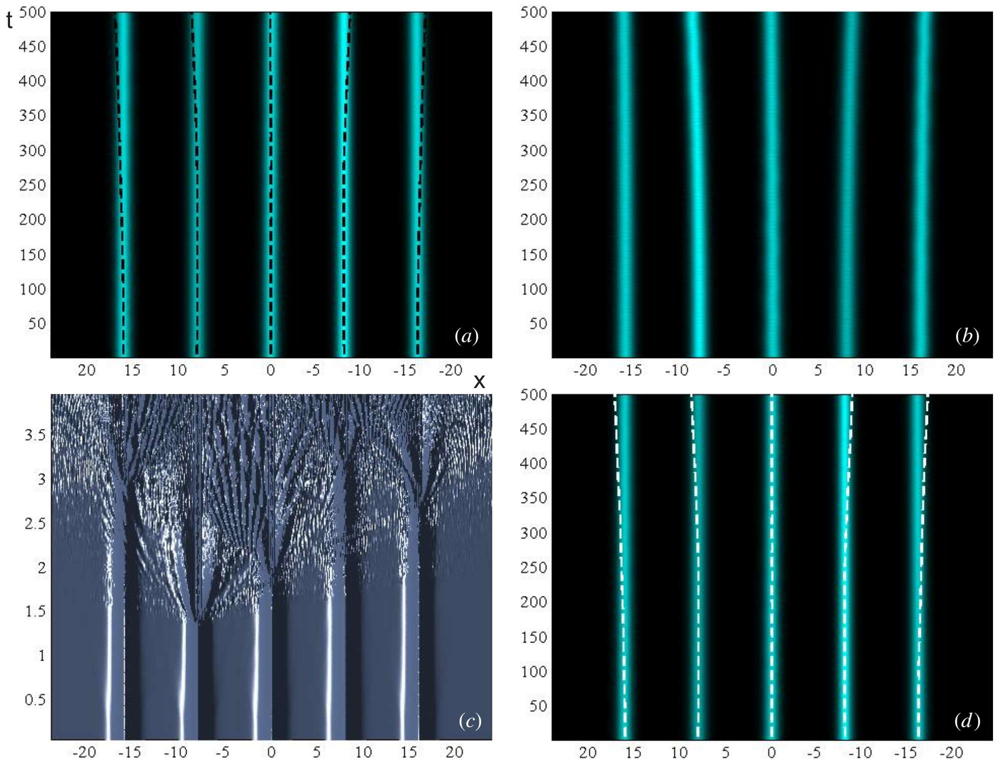

Figure 2. Dynamics of the five-soliton train: (*a*) quasi-equidistant propagation in a local CQ medium ($\gamma_3 = \gamma_5 = 0$), dashed lines correspond to the trajectories ξ_j, $j = 1, \ldots, 5$, obtained from equations (28), thick lines are solitons' tracks calculated from (3); (*b*) slightly disturbed quasi-equidistant propagation in a medium with only cubic nonlocality ($\gamma_3 = 0.0244, \gamma_5 = 0$); (*c*) chaotic disintegration of solitons in a medium with only quintic nonlocality ($\gamma_3 = 0, \gamma_5 = 0.06$); (*d*) restoration of the quasi-equidistant regime for nonlocality parameters obeying the condition (37) ($\gamma_3 = 0.0244$, $\gamma_5 = 0.06$). Initial configuration on all plots: $\beta = 0.05$, $\eta = 1/\sqrt{2}$, $V_j = 0$, $\varphi_{j,j-1} = \pi$, $\xi_j - \xi_{j-1} = 8$, $\delta = -0.1$.

Here the soliton power $P(\eta)$ is given by (8). Note that for the Gaussian response function (2) equation (37) can be rewritten in terms of nonlocality parameters $a_{3,5}$ as follows:

$$a_3 = \sqrt{\Gamma(\eta, \delta)} a_5.$$

Taking into account that γ_3 and γ_5 are positive, we have the restriction $\Gamma > 0$. Figure 3 shows the profile of the function Γ for the focusing quintic nonlinearity ($\delta > 0$). We observe that Γ is negative for any allowable choice of the parameters η and δ. Hence, the distortion compensation cannot be achieved in the case of the focusing quintic nonlinearity. At the same time, figure 4 shows that for negative values of δ there exists a region in the parameter space (δ, η) where $\Gamma > 0$ and a compensation of CQ nonlocalities can occur.

This prediction obtained from the generalized Toda system was checked by numerical simulation of (3) for defocusing quintic nonlinearity. Results are presented in figures 1(d) and 2(d). Here the dashed lines which show the solitons' trajectories calculated from equations (28) are compared with the solitons' tracks obtained by numerical integration of (3) for γ_3 and γ_5 obeying the condition (37). We see a good agreement between analytical and numerical results.

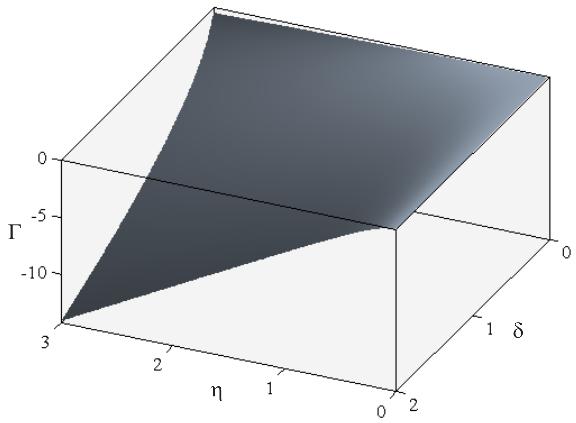

Figure 3. The profile of the Γ function for $\delta > 0$. Γ is negative for allowable soliton and medium parameters.

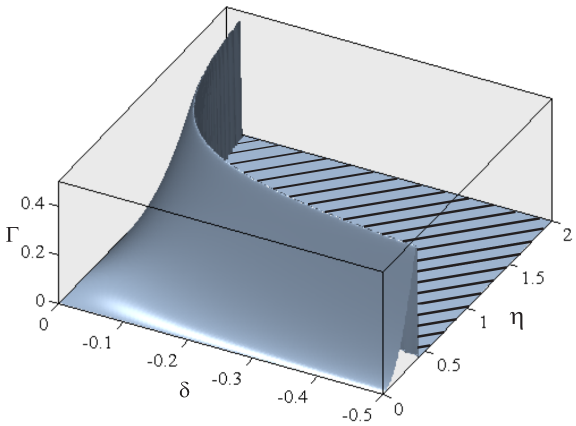

Figure 4. The profile of the Γ function for $\delta < 0$. The dashed area corresponds to the parameters' sets (δ, η) for which (7) violates and CQ solitons do not exist. There exists a domain of the parameters where Γ is positive, and cubic and quintic nonlocalities can be mutually compensated.

To investigate the effect of a departure from the exact relation (37), we fixed initial parameters and the soliton configuration (we put $\delta = -0.1$, $\eta = 1/\sqrt{2}$ and $\gamma_5 = 0.05$) and varied Γ. Very good compensation was observed within the interval $(1 \pm 0.08)\Gamma_0$, where Γ_0 is calculated according to (38) for the chosen parameters' set. Outside this interval, for $\Gamma < \Gamma_0$, chaotic behavior was detected. Increase of the contribution of the cubic nonlocality $(\Gamma > \Gamma_0)$ remedies the train dynamics by superseding disintegration with relatively stable propagation with enhanced variations of solitons peak amplitudes.

7. Conclusion

We have analyzed chainlike *N*-soliton dynamics in an essentially nonintegrable system governed by the CQ NLS equation. Using the multiple-scale perturbation approach, we have found evolution of soliton parameters on the slow time scale and reduced the above system to a generalized complex Toda chain model. Numerical simulations of the weakly nonlocal CQ NLS equation demonstrated that nonlocal responses can disturb a quasi-equidistant mode of the soliton train propagation. Moreover, a defocusing quintic nonlocality can drastically change soliton behavior leading to a quick development of chaotic disintegration of solitons.

From the Toda chain equation, we were able to predict a possibility of weakening chaotic manifestation in the soliton train dynamics, to the extent that a practically deterministic regime of the train propagation in a nonlocal medium can be restored. It should be stressed that such a restoration is achieved in a medium with the *defocusing* quintic nonlinearity only. Results of numerical integration of the CQ NLS equation are in a very good agreement with those derived from the Toda chain model.

We have studied a particular (cubic–quintic) type of nonlinearity. Such a choice is not too restrictive. Taking into account the results of [17], any local and, as it follows from our paper, any weakly nonlocal nonlinearity in the NLS-type equation will give the same evolution equations for the soliton parameters V, η and ξ_j, as in equation (28). The only place where the specific type of nonlinearity can be exhibited is the equation for the soliton phase σ_j.

References

[1] Takhtadjan L A and Faddeev L D 1984 *Hamiltonian Approach to Soliton Theory* (Berlin: Springer)
[2] Karpman V I and Solov'ev V V 1981 *Physica* D **3** 487
[3] Gorshkov K A and Ostrovsky L A 1981 *Physica* D **3** 428
[4] Arnold J M 1998 *J. Opt. Soc. Am.* A **15** 1450
[5] Arnold J M 1999 *Phys. Rev.* E **60** 979
[6] Gerdjikov V S, Kaup D J, Uzunov I M and Evstatiev E G 1996 *Phys. Rev. Lett.* **77** 3943
[7] Gerdjikov V S, Uzunov I M, Evstatiev E G and Diankov G L 1997 *Phys. Rev.* E **55** 6039
[8] Gerdjikov V S, Doktorov E V and Yang J 2001 *Phys. Rev.* E **64** 056617
[9] Doktorov E V, Matsuka N P and Rothos V M 2004 *Phys. Rev.* E **69** 056607
[10] Gerdjikov V S, Doktorov E V and Matsuka N P 2007 *Theor. Math. Phys.* **151** 773
[11] Shchesnovich V S 2002 *Phys. Rev.* E **65** 046614
[12] Yang J and Tan Y 2000 *Phys. Rev. Lett.* **85** 3624
[13] Tan Y and Yang J 2001 *Phys. Rev.* E **64** 056616
[14] Tan Y and Yang J 2001 *Phys. Lett.* A **280** 129
[15] Dmitriev S V, Kivshar Yu S and Shigenari T 2001 *Phys. Rev.* E **64** 056613
[16] Dmitriev S V and Shigenari T 2002 *Chaos* **12** 324
[17] Zhu Y and Yang J 2007 *Phys. Rev.* E **75** 036605
[18] Zhu Y, Haberman R and Yang J 2008 *Phys. Rev. Lett.* **100** 143901
[19] Krolikowski W, Bang O, Nikolov N J, Neshev D N, Rasmussen J J, Wyller J and Edmundson D 2004 *J. Opt. B: Quantum Semiclass. Opt.* **6** S288
[20] Yakimenko A I, Zaliznyak Yu A and Kivshar Yu S 2005 *Phys. Rev.* E **71** 065603
[21] Conti C, Peccianti M and Assanto G 2003 *Phys. Rev. Lett.* **91** 073901
[22] Conti C, Peccianti M and Assanto G 2004 *Phys. Rev. Lett.* **92** 113902
[23] Santos L, Shlyapnikov G V, Zoller P and Lewenstein M 2000 *Phys. Rev. Lett.* **85** 1791
[24] Perez-Garcia V M, Konotop V V and Garcia-Ripoll J J 2000 *Phys. Rev.* E **62** 4300
[25] Konotop V V and Perez-Garcia V M 2002 *Phys. Lett.* A **300** 348
[26] Altkorn R, Duval Malinsky M, Van Duyne R P and Koev T 2001 *Appl. Spectrosc.* **55** 373
[27] Cox F M, Argyros A and Large M C J 2006 *Opt. Express* **14** 4135
[28] Litvak A G and Sergeev A M 1978 *Pis'ma Zh. Eksp. Teor. Fiz.* **27** 547
Litvak A G and Sergeev A M 1978 *JETP Lett.* **27** 517
[29] Parola A, Salasnich L and Reatto L 1998 *Phys. Rev.* A **57** R3180
[30] Krolikowski W, Bang O, Rasmussen J J and Wyller J 2001 *Phys. Rev.* E **64** 016612
[31] Doktorov E V and Molchan M A 2007 *Phys. Rev.* A **75** 053819
[32] Mihalache D, Mazilu D, Lederer F, Crasovan L-C, Kartashov Y V, Torner L and Malomed B A 2006 *Phys. Rev.* E **74** 066614

[33] Pushkarov Kh I, Pushkarov D I and Tomov I V 1979 *Opt. Quantum Electron.* **11** 471
[34] Enns R H, Rangnekar S S and Kaplan A E 1987 *Phys. Rev.* A **35** 466
[35] Mihalache D, Mazilu D, Bertolotti M and Sibilia C 1988 *J. Opt. Soc. Am.* B **5** 565
[36] Gatz S and Herrmann J 1992 *IEEE J. Quantum Electron.* **28** 1732
[37] Khaykovich L and Malomed B A 2006 *Phys. Rev.* A **74** 023607
[38] Kaup D J 1990 *Phys. Rev.* A **42** 5689
[39] Pelinovsky D E, Kivshar Yu S and Afanasjev V V 1998 *Physica* D **116** 121
[40] Yang J and Kaup D J 2000 *SIAM J. Appl. Math.* **60** 967
[41] Yang J 2001 *Phys. Rev.* E **64** 026607